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ON SEMI-FREDHOLM PROPERTIES OF A BOUNDARY VALUE PROBLEM IN $\mathbf{R}^n_+$

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ABSTRACT
The paper considers a boundary value problem with the help of the smallest closed extension $\mathbf{L}^{\sim} : H_k \to H_{k_0} \times \mathscr{B}_{h_1} \times \cdots \times \mathscr{B}_{h_N}$ of a linear operator $\mathbf{L} : C^{\infty}_{(0)}(\mathbf{R}^n_+) \to \mathscr{S}(\mathbf{R}^n_+) \times \mathscr{S}(\mathbf{R}^{n-1}) \times \cdots \times \mathscr{S}(\mathbf{R}^{n-1})$. Here the spaces H_k (the spaces $\mathscr{B}_h$) are appropriate subspaces of $\mathscr{D}'(\mathbf{R}^n_+)$ (of $\mathscr{D}'(\mathbf{R}^{n-1})$, resp.), $\mathscr{S}(\mathbf{R}^n_+)$ and $C^{\infty}_{(0)}(\mathbf{R}^n_+)$) denotes the linear space of smooth functions $\mathbf{R}^n \to \mathbf{C}$, which are restrictions on $\mathbf{R}^n_+$ of a function from the Schwartz class $\mathscr{S}$ (from C^{∞}_0, resp.), $\mathscr{S}(\mathbf{R}^{n-1})$ is the Schwartz class of functions $\mathbf{R}^{n-1} \to \mathbf{C}$ and $\mathbf{L}$ is constructed by pseudo-differential operators. Criteria for the closedness of the range $R(\mathbf{L}^{\sim})$ and for the uniqueness of solutions $\mathbf{L}^{\sim} U = F$ are expressed. In addition, an *a priori* estimate for the corresponding boundary value problem is established.

1. Introduction

We consider semi-Fredholm properties of a non-elliptic boundary value system in $\mathbf{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_n \geqq 0\}$. The corresponding operators are assumed to be certain pseudo-differential operators. The spaces H_k (and $\mathscr{B}_h$), in which we are working, are subspaces of the distribution space $\mathscr{D}'(\mathbf{R}^n_+)$ (of $\mathscr{D}'(\mathbf{R}^{n-1})$). When the weight funtion k is chosen to be k_s, $s \in \mathbf{N}$, the space H_k is the totality of all the $L_2(\mathbf{R}^n_+)$-functions, whose distribution derivative $D^{\alpha}u$ also lies in $L_z(\mathbf{R}^n_+)$ for $|\alpha| \leqq s$ (here $k_s(\xi) = (1 + |\xi|^2)^{s/2}$).

When the local boundary value system is elliptic (for the terminology cf. [10] and [12]), the solutions of the corresponding boundary value problem satisfy some regularity properties and the solution operator obeys certain *a priori* estimates. In the case when the local elliptic boundary value problem is associated with an open bounded, sufficiently regular subset G of $\mathbf{R}^n$, the

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solution operator is a Fredholm operator (cf. [10] and [7], pp. 258–274, for example). These results can be extended for certain nonlocal elliptic boundary value problems ([12], [6], [4] and [11]). For related results of boundary value problems we refer to [3], [2] and [9], as well.

We shall deal with a (not necessarily local or elliptic) boundary value problem in the frames of the smallest closed extension $\mathbf{L}^{\sim} : H_k \to \mathbf{H}$ of a certain linear operator

$$\mathbf{L} : C^{\infty}_{(0)}(\mathbf{R}^n_+) \to \mathscr{S}(\mathbf{R}^n_+) \times \mathscr{S}(\mathbf{R}^{n-1}) \times \cdots \times \mathscr{S}(\mathbf{R}^{n-1}).$$

Here $\mathscr{S}(\mathbf{R}^n_+)$ denotes the totality of all smooth functions $\phi : \mathbf{R}^n \to \mathbf{C}$, where ϕ is the restriction on $\mathbf{R}^n_+$ of a function from the Schwartz class $\mathscr{S}$. $\mathscr{S}(\mathbf{R}^{n-1})$ is the Schwartz class corresponding the space $\mathbf{R}^{n-1}$. By

$$\mathbf{H} := H_{k_0} \times \mathscr{B}_{h_1} \times \cdots \times \mathscr{B}_{h_N}$$

we denote the product space which is associated with the given boundary value problem. We prove a sufficient condition for the surjectivity of the linear operator $\mathscr{L}$, which can be identified with the dual operator $\mathbf{L}^{\sim *}$ of $\mathbf{L}^{\sim}$ through linear homeomorphisms (cf. Theorem 4.2 and Corollary 4.4). Hence we obtain a criterion for the closedness of the range $R(\mathbf{L}^{\sim})$ and for the uniqueness of the solutions of $\mathbf{L}^{\sim} U = F$. This will finally lead us to the validity of a certain *a priori* estimate (cf. Corollary 4.5).

2. Definitions and notations

2.1. For the unexplained notions of the distribution theory and for the definition of the Hilbert spaces $\mathscr{B}_{2,k}$, $k \in \mathscr{K}$, we refer to [7]. The space $\mathscr{B}_{2,k}(\bar{\mathbf{R}}^n_-)$ is that closed subspace of $\mathscr{B}_{2,k}$ for whose element u it holds, supp $u \subset \bar{\mathbf{R}}^n_-$. Here we denoted

$$\mathbf{R}^n_- := \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_n < 0\}.$$

Similarly we denote $\mathbf{R}^n_+ := \{x \in \mathbf{R}^n \mid x_n > 0\}$. In the following we write $\mathscr{H}_k = \mathscr{B}_{2,k}$ and $\mathscr{H}_k(\bar{\mathbf{R}}^n_-) = \mathscr{B}_{2,k}(\bar{\mathbf{R}}^n_-)$.

The space $H_k^{\sim}(\mathbf{R}^n_+)$ is defined as a factor space

$$H_k^{\sim}(\mathbf{R}^n_+) = \mathscr{H}_k / \mathscr{H}_k(\bar{\mathbf{R}}^n_-) \tag{2.1}$$

equipped with the usual factor space topology induced by the norm

$$\| T \|_k^{\sim} = \inf_{u \in T} \| u \|_k \tag{2.2}$$

(here we denoted $\| u \|_k := \| u \|_{2,k}$).

Assume that $T \in H_k^{\sim}(\mathbf{R}_+^n)$ and that $u_T \in T$. Define a linear mapping $J : H_k^{\sim}(\mathbf{R}_+^n) \to \mathscr{D}'(\mathbf{R}_+^n)$ by $J(T) = u_T \big|_{\mathbf{R}_+^n}$. Then J is an injection. Let H_k be the subspace of $\mathscr{D}'(\mathbf{R}_+^n)$ given by $H_k = J(H_k^{\sim}(\mathbf{R}_+^n))$ equipped with the topology induced by the norm $\| V \|_k^+ := \| J^{-1}(V) \|_k^{\sim}$. Then a distribution $V \in \mathscr{D}'(\mathbf{R}_+^n)$ lies in H_k if and only if there exists $f_V \in \mathscr{H}_k$ such that

$$V(\phi) = f_V(\phi) \qquad \text{for all } \phi \in C_0^\infty(\mathbf{R}_+^n). \tag{2.3}$$

Let $C_{(0)}^\infty(\mathbf{R}_+^n)$ be the linear subspace of $C^\infty(\mathbf{R}_+^n)$ such that for each $\psi \in C_{(0)}^\infty(\mathbf{R}_+^n)$ there exists $f_\psi \in C_0^\infty$ with the property

$$\psi = f_\psi \big|_{\mathbf{R}_+^n}. \tag{2.4}$$

Then $C_{(0)}^\infty(\mathbf{R}_+^n)$ is dense in H_k (since C_0^∞ is dense in $\mathscr{H}_k$).

Finally the space $\mathscr{S}(\mathbf{R}_+^n)$ is defined as the (dense) subspace of H_k such that for each $\psi \in \mathscr{S}(\mathbf{R}_+^n)$ there exists $f_\psi \in \mathscr{S}$ with $\psi = f_\psi \big|_{\mathbf{R}_+^n}$.

Suppose that $V \in H_k$ and that $f_V \in \mathscr{H}_k$ with $V = f_V \big|_{\mathbf{R}_+^n}$. Then for all $\phi \in C_0^\infty(\mathbf{R}_+^n)$

$$|V(\phi)| = |f_V(\phi)| \leqq \| f_V \|_k \| \phi \|_{1/k^\vee},$$

where $k^\vee \in K$ such that $k^\vee(\xi) = k(-\xi)$. Hence one has

$$|V(\phi)| \leqq \| V \|_k^+ \| \phi \|_{1/k^\vee} \qquad \text{for all } \phi \in C_0^\infty(\mathbf{R}_+^n) \tag{2.5}$$

and then the topology of H_k is finer than the topology induced by $\mathscr{D}'(\mathbf{R}_+^n)$ on H_k.

2.2. Let $L(x, D)$ be a linear pseudo-differential operator on C_0^∞ with $C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$-symbol, that is, $L(x, D)$ is defined (under suitable temperating conditions about $L(x, \xi)$) by

$$(L(x, D)\phi)(x) := (2\pi)^{-n} \int_{\mathbf{R}^n} L(x, \xi)(\mathscr{F}\phi)(\xi)\, e^{i\langle \xi, x\rangle} d\xi \tag{2.6}$$

for $\phi \in C_0^\infty$ and $x \in \mathbf{R}^n$, where $\mathscr{F} : \mathscr{S}' \to \mathscr{S}'$ denotes the Fourier transform. We assume that $L(x, D)$ maps C_0^∞ into $\mathscr{S}$ and that the formal transpose $L'(x, D) : C_0^\infty \to \mathscr{S}$ of $L(x, D)$ exists, that is, there exists a linear operator $L'(x, D) : C_0^\infty \to \mathscr{S}$ such that

$$\begin{aligned} (L(x, D)\phi, \psi) &:= \int_{\mathbf{R}^n} (L(x, D)\phi)(x)\psi(x)dx \\ &= (\phi, L'(x, D)\psi) \qquad \text{for all } \phi, \psi \in C_0^\infty. \end{aligned} \tag{2.7}$$

For the sufficient algebraic criteria about the symbol $L(x,\xi)$ under which these assumptions hold we refer to [1].

Let $L(\cdot):\mathbf{R}^n\to\mathbf{C}$ be a C^∞-mapping such that for each $\alpha\in\mathbf{N}_0^n$ there exist $C_\alpha>0$ and $\mu_\alpha\in\mathbf{R}$ with

$$(2.8)\qquad |D_\xi^\alpha L(\xi)|\leqq C_\alpha(1+|\xi|^2)^{\mu_\alpha/2}=:C_\alpha k_{\mu_\alpha}(\xi).$$

Then the operator $L(D)$ defined by

$$(2.9)\qquad (L(D)\phi)(x)=(2\pi)^{-n}\int_{\mathbf{R}^n}L(\xi)(\mathcal{F}\phi)(\xi)e^{i(\xi,x)}d\xi$$

maps C_0^∞ into $\mathcal{S}$. The formal transpose $L'(D):C_0^\infty\to\mathcal{S}$ exists and

$$(2.10)\qquad (L'(D)\phi)(x)=(2\pi)^{-n}\int_{\mathbf{R}^n}L(-\xi)(\mathcal{F}\phi)(\xi)e^{i(\xi,x)}d\xi.$$

2.3. In the following we write $x=(x',x_n)$ for $x=(x_1,\dots,x_n)\in\mathbf{R}^n$. Let $\gamma_0:\mathcal{S}\to\mathcal{S}(\mathbf{R}^{n-1})$ (here we denote by $\mathcal{S}(\mathbf{R}^{n-1})$ the Schwartz class of functions $\mathbf{R}^{n-1}\to\mathbf{C}$) be a linear operator defined by

$$(2.11)\qquad (\gamma_0\phi)(x')=\phi(x',0)\qquad \text{for } x'\in\mathbf{R}^{n-1}$$

Furthermore, let $l_j(\cdot)$, $j=1,\dots,N$ be C^∞-mappings $\mathbf{R}^n\to\mathbf{C}$ such that

$$(2.12)\qquad |(D_\xi^\alpha l_j)(\xi)|\leqq C_\alpha k_{\mu_\alpha}(\xi)\qquad \text{for all } \xi\in\mathbf{R}^n.$$

Then the corresponding pseudo-differential operators $l_j(D)$ map (as we mentioned above) C_0^∞ into $\mathcal{S}$ and the formal transposes $l_j'(D):C_0^\infty\to\mathcal{S}$ exist.

Denote by $\mathcal{K}(\mathbf{R}^{n-1})$ the class of weight functions $h:\mathbf{R}^{n-1}\to\mathbf{R}$ defined in the same way as the class $\mathcal{K}$ of weight functions $k:\mathbf{R}^n\to\mathbf{R}$. Let h be in $\mathcal{K}(\mathbf{R}^{n-1})$. The spaces $\mathcal{B}_h$ are defined (as the corresponding spaces $\mathcal{H}_k$ in $\mathcal{S}'$) as the totality of all tempered distributions $u\in\mathcal{S}'(\mathbf{R}^{n-1})$ for which $\mathcal{F}_{n-1}u$ lies in $L_1^{\mathrm{loc}}(\mathbf{R}^{n-1})$ and

$$(2.13)\qquad \|u\|_h:=\left((2\pi)^{-(n-1)}\int_{\mathbf{R}^{n-1}}|(\mathcal{F}_{n-1}u)(\xi')h(\xi')|^2d\xi'\right)^{1/2}<\infty.$$

Here $\mathcal{F}_{n-1}$ denotes the Fourier transform $\mathcal{S}'(\mathbf{R}^{n-1})\to\mathcal{S}'(\mathbf{R}^{n-1})$. The topology in $\mathcal{B}_h$ is that of induced by the norm (2.13). One sees that $\mathcal{B}_h$ is also a Hilbert space.

2.4. Choose k and k_0 from $\mathcal{K}$ and h_j, $j=1,\dots,N$ from $\mathcal{K}(\mathbf{R}^{n-1})$. The product space (equipped with the standard product space topology) $H_{k_0}\times\mathcal{B}_{h_1}\times\cdots\times\mathcal{B}_{h_N}$ is denoted by $\mathbf{H}$.

Let $L(x, D)$ and $l_j(D)$ be as in the Sections 2.2 and 2.3 with the following additional property:

$$(2.14)\qquad \begin{cases} L'(x, D)\theta \text{ lies in } C_0^\infty(\mathbf{R}^n_+) \text{ when } \theta \text{ lies in } C_0^\infty(\mathbf{R}^n_+), \\ l_j'(D)\theta \text{ and } l_j(D)\theta \text{ lie in } C_0^\infty(\mathbf{R}^n_+) \text{ when } \theta \text{ lies in } C_0^\infty(\mathbf{R}^n_+). \end{cases}$$

As is well-known, the condition (2.14) holds when $L'(x, D)$, $l_j(D)$ and $l_j'(D)$ are so-called properly supported in $\mathbf{R}^n_+$ (cf. [13], p. 43).

Define a dense linear operator $\mathbf{L}: H_k \to \mathbf{H}$ with

$$(2.15)\qquad \begin{cases} D(\mathbf{L}) = C_{(0)}^\infty(\mathbf{R}^n_+), \\ \mathbf{L}\phi = (L\phi, l_1\phi, \dots, l_N\phi) \qquad \text{for } \phi \in D(\mathbf{L}), \end{cases}$$

where

$$(2.16)\qquad \begin{cases} L\phi = (L(x, D) f_\phi)\,|_{\mathbf{R}^n_+}, \\ l_j\phi = \gamma_0(L(D) f_\phi). \end{cases}$$

Here f_ϕ lies in C_0^∞ such that $\phi = f_\phi\,|_{\mathbf{R}^n_+}$. The operator $\mathbf{L}$ is well-defined: Suppose that $\phi = \psi$. Then due to (2.14) for all $\theta \in C_0^\infty(\mathbf{R}^n_+)$ one has

$$(2.17)\qquad \begin{aligned} (L(x, D) f_\phi, \theta) &= (f_\phi, L'(x, D)\theta) = (f_\psi, L'(x, D)\theta) \\ &= (L(x, D) f_\psi, \theta), \end{aligned}$$

and then $L\phi = L\psi$. Similarly one sees that $l_j\phi = l_j\psi$, since by (2.14) $l_j(D) f_\phi = l_j(D) f_\psi$ in $\bar{\mathbf{R}}^n_+$. Hence $\mathbf{L}\phi = \mathbf{L}\psi$.

Furthermore we have

Lemma 2.1. *Suppose that there exist $q \in \mathbf{N}$, $\varepsilon > 0$ and $C > 0$ such that for all $\xi = (\xi', \xi_n) \in \mathbf{R}^n$ and $j = 1, \dots, N$ one has*

$$(2.18)\qquad l_j(\xi) k_{-q}(\xi') k_{(1/2)+\varepsilon}(\xi_n) \leqq C k(\xi).$$

Then the operator $\mathbf{L}: H_k \to \mathbf{H}$ is closable.

Proof. Let $\{\phi_m\} \subset C_{(0)}^\infty(\mathbf{R}^n_+)$ be a sequence such that with $F = (f_0, g_1, \dots, g_N) \in \mathbf{H}$ one has

$$(2.19)\qquad \begin{cases} \|\phi_m\|_k^+ \to 0 \qquad \text{with } m \to \infty, \\ \|\mathbf{L}\phi_m - F\| := \|L\phi_m - f_0\|_{k_0}^+ + \sum_{j=1}^N \|l_j\phi_m - g_j\|_{h_j} \to 0. \end{cases}$$

We have to show that $F=0$.

For all $\theta\in C_0^\infty(\mathbf{R}^n_+)$ we obtain by (2.5) and (2.14)

$$f_0(\theta)=\lim_{m\to\infty}(L\phi_m,\theta)=\lim_{m\to\infty}(\phi_m,L'(x,D)\theta)=0 \tag{2.20}$$

(since $(L\phi_m,\theta)=(L(x,D)f_{\phi_m},\theta)=(f_{\phi_m},L'(x,D)\theta)=(\phi_m,L'(x,D)\theta)$). Hence we have $f_0=0$.

Since $C_0^\infty(\mathbf{R}^n_-)$ is dense in $\mathscr{H}_k(\bar{\mathbf{R}}^n_-)$ (which is easy to see by using the standard cutting and regularizing process; cf. also [7], p. 52) we can choose a sequence $\{\psi_m\}\subset C_0^\infty(\mathbf{R}^n_-)$ such that

$$\| f_{\phi_m}+\psi_m\|_k\to 0. \tag{2.21}$$

For all $(\xi',\xi_n)\in\mathbf{R}^n$ and $\phi\in\mathscr{S}$ one has

$$\begin{aligned}\mathscr{F}_{n-1}(\gamma_0\phi)(\xi')&=\int_{\mathbf{R}^{n-1}}\phi(x',0)e^{-i(\xi',x')}dx'\\&=(2\pi)^{-1}\int_{\mathbf{R}}\mathscr{F}_1\left(\int_{\mathbf{R}^{n-1}}\phi(x',\cdot)e^{-i(\xi',x')}dx'\right)(t)dt\\&=(2\pi)^{-1}\int_{\mathbf{R}}(\mathscr{F}\phi)(\xi)d\xi_n.\end{aligned} \tag{2.22}$$

where we used the Fourier inversion formula (here $\mathscr{F}_1$ denotes the Fourier transform $\mathscr{S}'(\mathbf{R})\to\mathscr{S}'(\mathbf{R})$). Applying (2.21)–(2.22) one sees finally that for all $\theta'\in C_0^\infty(\mathbf{R}^{n-1})$

$$\begin{aligned}|g_j(\theta')|&=\lim_{m\to\infty}|(l_j\phi_m,\theta')|\\&=\lim_{m\to\infty}\left|(2\pi)^{-(n-1)}\int_{\mathbf{R}^{n-1}}\mathscr{F}_{n-1}(l_j\phi_m)(\xi')\overline{\mathscr{F}_{n-1}(\overline{\theta'})}(\xi')d\xi'\right|\\&=\lim_{m\to\infty}\left|(2\pi)^{-(n-1)}\int_{\mathbf{R}^{n-1}}\mathscr{F}_{n-1}(\gamma_0(l_j(D)(f_{\phi_m}+\psi_m)))(\xi')\overline{(\mathscr{F}_{n-1}\overline{\theta'})}(\xi')d\xi'\right|\\&=\left|\lim_{m\to\infty}(2\pi)^{-n}\int_{\mathbf{R}^n}l_j(\xi)\mathscr{F}(f_{\phi_m}+\psi_m)(\xi)\overline{(\mathscr{F}_{n-1}\overline{\theta'})}(\xi')d\xi'\right|\\&\leqq\lim_{m\to\infty}(2\pi)^{-n}\int_{\mathbf{R}^n}\left|l_j(\xi)\mathscr{F}(f_{\phi_m}+\psi_m)(\xi)(\mathscr{F}_{n-1}\overline{\theta'})(\xi')\right|d\xi'.\end{aligned} \tag{2.23}$$

Here we used the fact that $l_j(D)\phi \in C_0^\infty(\mathbf{R}^n_-)$ for $\phi \in C_0^\infty(\mathbf{R}^n_-)$, which can be seen as follows: For all $\phi \in C_0^\infty(\mathbf{R}^n_-)$

$$
\begin{aligned}
(l_j(D)\phi)(x) &= (2\pi)^{-n} \int_{\mathbf{R}^n} l_j(\xi)(\mathscr{F}\phi)(\xi) e^{i(\xi,x)} d\xi \\
&= (2\pi)^{-n} \int_{\mathbf{R}^n} l_j(-\xi)(\mathscr{F}\phi^\vee)(\xi) e^{i(\xi,-x)} d\xi \qquad (2.24)\\
&= (l_j'(D)\phi^\vee)^\vee(x)
\end{aligned}
$$

and then by (2.14) $l_j(D)\phi$ lies in $C_0^\infty(\mathbf{R}^n_-)$ for $\phi \in C_0^\infty(\mathbf{R}^n_-)$.

Using the assumption (2.18) we obtain by (2.23)

$$
\begin{aligned}
|g_j(\theta')| &\leqq (2\pi)^{-n} C \int_{\mathbf{R}^n} k_{-((1/2)+\varepsilon)}(\xi_n) k_q(\xi') k(\xi) \\
&\qquad \cdot |\mathscr{F}(f_{\phi_m} + \psi_m)(\xi)(\mathscr{F}_{n-1}\overline{\theta'})(\xi')| \, d\xi' \qquad (2.25)\\
&\leqq (2\pi)^{-n} C \left(\int_{\mathbf{R}^n} (1/((1+|\xi_n|^2)^{((1/2)+\varepsilon)})) |(\mathscr{F}_{n-1}\overline{\theta'})(\xi') k_q(\xi')|^2 d\xi \right)^{1/2} \\
&\qquad \cdot \left(\int_{\mathbf{R}^n} |\mathscr{F}(f_{\phi_m} + \psi_m)(\xi) k(\xi)|^2 d\xi \right)^{1/2} \to 0 \qquad \text{with } m \to \infty,
\end{aligned}
$$

and then $g_j(\theta') = 0$ for all $\theta' \in C_0^\infty(\mathbf{R}^{n-1})$, that is, $g_j = 0$. Hence the proof is ready. □

Let $\mathbf{L}^\sim : H_k \to \mathbf{H}$ be the smallest closed extension of $\mathbf{L}$, that is, $u \in D(\dot{\mathbf{L}}^\sim)$ if and only if there exists a sequence $\{\phi_m\} \subset D(\mathbf{L})$ such that with some $F \in \mathbf{H}$

$$\| \phi_m - u \|_k^+ \to 0 \qquad \text{with } m \to \infty$$

and

$$\| \mathbf{L}\phi_m - F \| \to 0 \qquad \text{with } m \to \infty.$$

We now list the conditions which shall be assumed in the sequel:

1° $L(x, D)$ is a linear pseudo-differential operator $C_0^\infty \to \mathscr{S}$ such that the formal transpose $L'(x, D) : C_0^\infty \to \mathscr{S}$ exists.

2° $l_j(D), j = 1, \ldots, N$ are linear pseudo-differential operators $C_0^\infty \to \mathscr{S}$ with the symbol $l_j(\xi)$, where $l_j(\xi)$ obeys (2.12).

3° $L'(x, D)$, $l_j(D)$ and $l_j'(D), j = 1, \ldots, N$ satisfy the property (2.14).

4° The weight function $k \in \mathscr{K}$ and the mappings $l_j(\cdot)$ satisfy (2.18).

3. On the solvability of the dual equation $\mathbf{L}^{\sim *}U = F$

3.1. Let k and k_0 be in $\mathscr{K}$ and let h_j, $j = 1, \dots, N$ be in $\mathscr{K}(\mathbf{R}^{n-1})$. In this section we suppose the assumptions 1°–4° of Section 2 (without any particular mention). Then we can form the minimal closed extension $\mathbf{L}^{\sim} : H_k \to \mathbf{H}$ of $\mathbf{L}$. In the sequel some semi-Fredholm properties of $\mathbf{L}^{\sim}$ are considered.

Let $\mathscr{H}_k(\mathbf{R}^n_+)$ be the completion of $C_0^\infty(\mathbf{R}^n_+)$ in $\mathscr{H}_k$. Then we have $\mathscr{H}_k(\mathbf{R}^n_+) = \mathscr{H}_k(\bar{\mathbf{R}}^n_+)$, where $\mathscr{H}_k(\bar{\mathbf{R}}^n_+)$ is the subspace of $\mathscr{H}_k$, whose elements u satisfy, $\operatorname{supp} u \subset \bar{\mathbf{R}}^n_+$.

We begin with the following lemma, which reveals the structure of the dual space $\mathbf{H}^*$ of $\mathbf{H} = H_{k_0} \times \mathscr{B}_{h_1} \times \cdots \times \mathscr{B}_{h_N}$.

LEMMA 3.1. *Assume that T is in $\mathbf{H}^*$. Then there exists $t = (t_0, t_1, \dots, t_N) \in \mathscr{H}_{1/k_0^\vee}(\mathbf{R}^n_+) \times \mathscr{B}_{1/h_1^\vee} \times \cdots \times \mathscr{B}_{1/h_N^\vee}$ such that*

$$T(\Phi) = t_0(f_\phi) + \sum_{j=1}^{N} t_j(\theta_j) \tag{3.1}$$

for all $\Phi = (\phi, \theta_1, \dots, \theta_N) \in \mathscr{S}(\mathbf{R}^n_+) \times \mathscr{S}(\mathbf{R}^{n-1}) \times \cdots \times \mathscr{S}(\mathbf{R}^{n-1})$, where $f_\phi \in \mathscr{S}$ with $f_\phi|_{\mathbf{R}^n_+} = \phi$.

Conversely, suppose that $t = (t_0, t_1, \dots, t_N)$ lies in

$$\mathscr{H}_{1/k_0^\vee}(\mathbf{R}^n_+) \times \mathscr{B}_{1/h_1^\vee} \times \cdots \times \mathscr{B}_{1/h_N^\vee}.$$

Then the linear form $L : \mathscr{S}(\mathbf{R}^n_+) \times \mathscr{S}(\mathbf{R}^{n-1}) \times \cdots \times \mathscr{S}(\mathbf{R}^{n-1}) \to \mathbf{C}$ such that $L(\Phi) = t_0(f_\phi) + \sum_{j=1}^N t_j(\theta_j)$ can be continuously extended onto $\mathbf{H}$.

PROOF. (A) Suppose that T lies in $\mathbf{H}^*$. then for all $W = (w_0, w_1, \dots, w_N) \in \mathbf{H}$ one has

$$T(w) = T(w_0, 0, \dots, 0) + \sum_{j=1}^{N} T(0, \dots, w_j, \dots, 0). \tag{3.2}$$

The functional $T_0 : w_0 \to T(w_0, 0, \dots, 0)$ is bounded in H_{k_0} and the functionals $T_j : w_j \to T(0, \dots, 0, w_j, 0, \dots, 0)$ are bounded in $\mathscr{B}_{h_j}$. Hence one sees that T can be written in the form

$$T(W) = T_0(w_0) + \sum_{j=1}^{N} T_j(w_j), \tag{3.3}$$

where $T_0 \in H^*_{k_0}$ and $T_j \in \mathscr{B}^*_{h_j}$.

(B) For each $T_j \in \mathscr{B}^*_{h_j}$ there exists $t_j \in \mathscr{B}_{1/h_j^\vee}$ such that

$$T_j\theta = t_j(\theta) \qquad \text{for all } \theta \in \mathscr{S}(\mathbf{R}^{n-1}) \tag{3.4}$$

(cf. [7], p. 43). We will show that for each $T_0 \in H_{k_0}^*$ there exists $t_0 \in \mathscr{H}_{1/k_0^\vee}(\mathbf{R}_+^n)$ such that

$$T_0(\phi) = t_0(f_\phi) \qquad \text{for all } \phi \in \mathscr{S}(\mathbf{R}_+^n). \tag{3.5}$$

Let T_0 be in $H_{k_0}^*$; then for all $\psi \in \mathscr{S}$

$$|T_0(\psi|_{\mathbf{R}_+^n})| \leqq \| T_0 \| \; \| \psi|_{\mathbf{R}_+^n} \|_{k_0}^+ \leqq \| T_0 \| \; \| \psi \|_{k_0}. \tag{3.6}$$

Hence there exists $t_0 \in \mathscr{H}_{1/k_0^\vee}$ such that

$$T_0(\psi|_{\mathbf{R}_+^n}) = t_0(\psi) \qquad \text{for all } \psi \in \mathscr{S} \tag{3.7}$$

(cf. [7], p. 43). Since $t_0(\psi) = 0$ for all $\psi \in C_0^\infty(\mathbf{R}_-^n)$ one sees that t_0 lies in $\mathscr{H}_{1/k_0^\vee}(\bar{\mathbf{R}}_+^n) = \mathscr{H}_{1/k_0^\vee}(\mathbf{R}_+^n)$. Combining relations (3.3), (3.4) and (3.7) we obtain

$$T(\Phi) = t_0(f_\phi) + \sum_{j=1}^{N} t_j(\theta_j)$$

for all $\Phi = (\phi, \theta_1, \ldots, \theta_N) \in \mathscr{S}(\mathbf{R}_+^n) \times \mathscr{S}(\mathbf{R}^{n-1}) \times \cdots \times \mathscr{S}(\mathbf{R}^{n-1})$, as required.

(C) Conversely, we assume that $t = (t_0, t_1, \ldots, t_N)$ is in $\mathscr{H}_{1/k_0^\vee}(\mathbf{R}_+^n) \times \mathscr{B}_{1/h_1^\vee} \times \cdots \times \mathscr{B}_{1/h_N^\vee}$. Then the linear form L given in the assertion is well-defined, since the relation $\phi_1 = \phi_2 \in \mathscr{S}(\mathbf{R}_+^n)$ implies that $\operatorname{supp}(f_{\phi_1} - f_{\phi_2}) \subset \bar{\mathbf{R}}_-^n$ (note that $C_0^\infty(\mathbf{R}_+^n)$ is dense in $\mathscr{H}_{1/k_0^\vee}(\mathbf{R}_+^n)$). We show that

$$|t_0(f_\phi)| \leqq \| t_0 \|_{1/k_0^\vee} \| \phi \|_{k_0}^+ \qquad \text{for all } \phi \in \mathscr{S}(\mathbf{R}_+^n). \tag{3.8}$$

In fact we obtain for all $\psi \in C_0^\infty(\mathbf{R}_-^n)$

$$|t_0(f_\phi)| = |t_0(f_\phi + \psi)| \leqq \| t_0 \|_{1/k_0^\vee} \| f_\phi + \psi \|_{k_0} \tag{3.9}$$

and then (3.8) is valid. In virtue of (3.8) we get for all $\Phi = (\phi, \theta_1, \ldots, \theta_N) \in \mathscr{S}(\mathbf{R}_+^n) \times \mathscr{S}(\mathbf{R}^{n-1}) \times \cdots \times \mathscr{S}(\mathbf{R}^{n-1})$

$$\begin{aligned} |L\phi| &\leqq \| t_0 \|_{1/k_0^\vee} \| \phi \|_{k_0}^+ + \sum_{j=1}^{N} \| t_j \|_{1/h_j^\vee} \| \theta_j \|_{h_j} \\ &\leqq \left(\| t_0 \|_{1/k_0^\vee} + \sum_{j=1}^{N} \| t_j \|_{1/h_j^\vee} \right) \| \Phi \| \end{aligned} \tag{3.10}$$

(in the product space $\mathbf{H}$ we use the sum-norm $\| \phi \|_{k_0}^+ + \Sigma_{j=1}^N \| \theta_j \|_{h_j}$). Hence the proof is complete. □

In virtue of Lemma 3.1 the linear mapping $\lambda : \mathbf{H}^* \to \mathscr{H}_{1/k_0^\vee}(\mathbf{R}^n_+) \times \mathscr{B}_{1/h_1^\vee} \times \cdots \times \mathscr{B}_{1/h_N^\vee}$ defined by

$$\lambda(T) = (t_0, t_1, \ldots, t_N) \tag{3.11}$$

is a bijection, since $\mathscr{S}(\mathbf{R}^n_+)$ is dense in H_{k_0} and $\mathscr{S}(\mathbf{R}^{n-1})$ is dense in $\mathscr{B}_{1/h_j^\vee}$.

Lemma 3.2. *The mapping* $\lambda : \mathbf{H}^* \to \mathscr{H}_{1/k_0^\vee}(\mathbf{R}^n_+) \times \mathscr{B}_{1/h_1^\vee} \times \cdots \times \mathscr{B}_{1/h_N^\vee}$ *given by* (3.11) *is a linear homeomorphism.*

Proof. λ is a bijection and by (3.10)

$$\| T \| \leqq \| t_0 \|_{1/k_0^\vee} + \sum_{j=1}^{N} \| t_j \|_{1/h_j^\vee} = \| \lambda(T) \| . \tag{3.12}$$

On the other hand, for all $\psi \in \mathscr{S}$ and $\theta_j \in \mathscr{S}(\mathbf{R}^{n-1})$

$$\begin{aligned} | t_0(\psi) | = | T(\psi |_{\mathbf{R}^n_+}, 0, \ldots, 0) | &\leqq \| T \| \ \| \psi |_{\mathbf{R}^n_+} \|^+_{k_0} \\ &\leqq \| T \| \ \| \psi \|_{k_0} \end{aligned} \tag{3.13}$$

and

$$| t_j(\theta_j) | = | T(0, \ldots, \theta_j, \ldots, 0) | \leqq \| T \| \ \| \theta_j \|_{h_j}. \tag{3.14}$$

Hence $\| t_0 \|_{1/k_0^\vee} \leqq \| T \|$ and $\| t_j \|_{1/h_j^\vee} \leqq \| T \|$ (cf. [7], p. 43), which implies that

$$\| \lambda(T) \| \leqq (N+1) \| T \| . \tag{3.15}$$

This proves the Lemma. □

As the proofs of the previous Lemmas show, the dual space H_k^* of H_k can be characterized in the following way:

Lemma 3.3. *Assume that F is in H_k^*. Then there exists $f \in \mathscr{H}_{1/k^\vee}(\mathbf{R}_+)$ such that*

$$F(\psi |_{\mathbf{R}^n_+}) = f(\psi) \qquad \textit{for all } \psi \in \mathscr{S}. \tag{3.16}$$

On the other hand, let f be in $\mathscr{H}_{1/k^\vee}(\mathbf{R}^n_+)$. Then the linear form $L : \mathscr{S}(\mathbf{R}^n_+) \to \mathbf{C}$ given by $L\phi = f(f_\phi)$ has the continuous extension onto H_k.

Furthermore, the linear bijection $\kappa : H_k^* \to \mathscr{H}_{1/k^\vee}(\mathbf{R}^n_+)$ such that $\kappa(F) = f$ is an isometrical isomorphism.

3.2. Define a linear operator $\mathscr{L} : \mathscr{H}_{1/k_0^\vee}(\mathbf{R}^n_+) \times \mathscr{B}_{1/h_1^\vee} \times \cdots \times \mathscr{B}_{1/h_N^\vee} \to \mathscr{H}_{1/k^\vee}(\mathbf{R}^n_+)$ by the requirement

$$(3.17)\quad \begin{cases} D(\mathscr{L}) = \Big\{ u = (u_0, u_1 \dots, u_N) \in \mathscr{H}_{1/k_0^\vee}(\mathbf{R}^n_+) \times \mathscr{B}_{1/h_1^\vee} \times \cdots \times \mathscr{B}_{1/h_N^\vee} \,\Big| \\ \qquad \text{there exists } f \in \mathscr{H}_{1/k^\vee}(\mathbf{R}^n_+) \text{ such that} \\ \qquad u_0(L(x,D)\phi) + \sum_{j=1}^{N} u_j(\gamma_0(l_j(D)\phi)) = f(\phi) \text{ for } \phi \in C_0^\infty \Big\}, \\ \mathscr{L}u = f. \end{cases}$$

$\mathscr{L}$ is closed and (by 1°–4°) densily defined. The connection between operator $\mathscr{L}$, just defined, and the dual operator $\mathbf{L}^{\sim *}: \mathbf{H}^* \to H_k^*$ is given by

Theorem 3.4. *Let $\lambda: \mathbf{H}^* \to \mathscr{H}_{1/k_0^\vee}(\mathbf{R}^n_+) \times \mathscr{B}_{1/h_1^\vee} \times \cdots \times \mathscr{B}_{1/h_N^\vee}$ and $\kappa: H_k^* \to \mathscr{H}_{1/k^\vee}(\mathbf{R}^n_+)$ be the linear homeomorphisms given in Section* 3.1. *Then one has*

$$(3.18)\qquad \mathscr{L} = \kappa \circ \mathbf{L}^{\sim *} \circ \lambda^{-1}.$$

Proof. (A) Assume that $U \in D(\mathbf{L}^{\sim *})$ and $\mathbf{L}^{\sim *}U = F$, that is, $U \in \mathbf{H}^*$ such that

$$(3.19)\qquad U(\mathbf{L}^{\sim} v) = Fv \qquad \text{for all } v \in D(\mathbf{L}^{\sim})$$

with some $F \in H_k^*$. Then due to Lemma 3.1

$$(3.20)\qquad U(\psi\,|_{\mathbf{R}^n_+}, \theta_1, \dots, \theta_N) = u_0(\psi) + \sum_{j=1}^{N} u_j(\theta_j)$$

for all $(\psi, \theta_1 \dots, \theta_N) \in \mathscr{S} \times \mathscr{S}(\mathbf{R}^{n-1}) \times \cdots \times \mathscr{S}(\mathbf{R}^{n-1})$, where $(u_0, u_1, \dots, u_N) = \lambda U$. Hence for all $\psi \in C_0^\infty$

$$(3.21)\qquad \begin{aligned} U(\mathbf{L}(\psi\,|_{\mathbf{R}^n_+})) &= U((L(x,D)\psi)\,|_{\mathbf{R}^n_+}, \gamma_0(l_1(D)\psi), \dots, \gamma_0(l_N(D)\psi)) \\ &= u_0(L(x,D)\psi) + \sum_{j=1}^{N} u_j(\gamma_0(l_j(D)\psi)). \end{aligned}$$

Similarly one sees that for all $\psi \in C_0^\infty$

$$(3.22)\qquad F(\psi\,|_{\mathbf{R}^n_+}) = (\kappa F)(\psi) := f(\psi),$$

where $f = \kappa F$. Since by (3.19)

$$(3.23)\qquad U(\mathbf{L}(\psi\,|_{\mathbf{R}^n_+})) = F(\psi\,|_{\mathbf{R}^n_+}) \qquad \text{for all } \psi \in C_0^\infty,$$

we get from (3.21)–(3.23) that $\lambda U \in D(\mathscr{L})$ and that $\mathscr{L}(\lambda U) = f = \kappa F$. This shows that $D(\kappa \circ \mathbf{L}^{\sim *} \circ \lambda^{-1}) \subset D(\mathscr{L})$ and that $\mathscr{L}u = (\kappa \circ \mathbf{L}^{\sim *} \circ \lambda^{-1})u$ for all $u \in D(\kappa \circ \mathbf{L}^{\sim *} \circ \lambda^{-1})$.

(B) Let u be in $D(\mathscr{L})$ and let $\mathscr{L}u = f$. Then for all $\phi \in C^\infty_{(0)}(\mathbf{R}^n_+)$

$$(\lambda^{-1}u)(\mathbf{L}\phi) = (\lambda^{-1}u)((L(x,D)f_\phi)\,|_{\mathbf{R}^n_+}, \gamma_0(l_1(D)f_\phi), \ldots, \gamma_0(l_N(D)f_\phi))$$

$$= u_0(L(x,D)f_\phi) + \sum_{j=1}^{N} u_j(\gamma_0(l_j(D)f_\phi))$$

$$= f(f_\phi) \tag{3.24}$$

$$= (\kappa^{-1}f)(\phi).$$

Thus for all $v \in D(\mathbf{L}^\sim)$

$$(\lambda^{-1}u)(\mathbf{L}^\sim v) = (\kappa^{-1}f)(v),$$

that is, $\lambda^{-1}u \in D(\mathbf{L}^{\sim *})$ and $\mathbf{L}^{\sim *}(\lambda^{-1}u) = \kappa^{-1}f$. This shows that $D(\mathscr{L}) \subset D(\kappa \circ \mathbf{L}^{\sim *} \circ \lambda^{-1})$ and so the proof is ready. □

Since κ and λ are linear homeomorphisms and since the range $R(\mathbf{L}^{\sim *})$ is closed if and only if the range $R(\mathbf{L}^\sim)$ is closed (for the general theory of closed dense operators cf. [8], pp. 163–236), one obtains

Corollary 3.5. *The range $R(\mathbf{L}^\sim)$ is closed in* $\mathbf{H}$ *if and only if the range $R(\mathscr{L})$ is closed in $\mathscr{H}_{1/k^\vee}(\mathbf{R}^n_+)$.*

Furthermore, it is easy to see

Corollary 3.6. *Suppose that $R(\mathscr{L})$ is closed. Then one has*

$$\dim N(\mathbf{L}^\sim) = \operatorname{codim} R(\mathscr{L}), \tag{3.25}$$

$$\dim N(\mathscr{L}) = \operatorname{codim} R(\mathscr{L}^\sim) \tag{3.26}$$

and

$$\operatorname{ind}(\mathbf{L}^\sim) := \dim N(\mathbf{L}^\sim) - \operatorname{codim} R(\mathbf{L}^\sim) = -\operatorname{ind}(\mathscr{L}). \tag{3.27}$$

Here $N(\mathbf{L}^\sim)$ (and $N(\mathscr{L})$) presents the kernel of $\mathbf{L}^\sim$ (the kernel of $\mathscr{L}$, resp.).

Combining Corollaries 3.5 and 3.6 one sees that $\mathbf{L}^\sim$ is a (semi-)Fredholm operator if and only if $\mathscr{L}$ is a (semi-)Fredholm operator. The following Corollary is also obvious

Corollary 3.7. *Suppose that $R(\mathscr{L})$ is closed. Then the relation*

$$R(\mathscr{L}) = \mathscr{H}_{1/k^\vee}(\mathbf{R}^n_+) \tag{3.28}$$

is true if and only if

$$N(\mathbf{L}^{\sim}) = \{0\}. \tag{3.29}$$

Similarly, the relation

$$R(\mathbf{L}^{\sim}) = \mathbf{H} \tag{3.30}$$

is true if and only if

$$N(\mathscr{L}) = \{0\}. \tag{3.31}$$

3.3. The existence of solutions $\mathscr{L}u = f$ can be characterized in the following way:

THEOREM 3.8. *Let $u = (u_0, u_1, \ldots, u_N)$ be in $\mathscr{H}_{1/k_0^\vee}(\mathbf{R}^n_+) \times \mathscr{B}_{1/h_1^\vee} \times \cdots \times \mathscr{B}_{1/h_N^\vee}$ and let f be in $\mathscr{H}_{1/k^\vee}(\mathbf{R}^n_+)$. Then u lies in $D(\mathscr{L})$ and $\mathscr{L}u = f$ if and only if $u_0 \in D(L^\#)$ and one has*

$$\mathscr{F}(L^\# u_0 - f)(\xi) + \sum_{j=1}^{N} (2\pi)^{-1} l_j(-\xi)(\mathscr{F}_{n-1}u_j)(\xi') = 0, \tag{3.32}$$

a.e. $\xi = (\xi', \xi_n) \in \mathbf{R}^n$. Here the operator

$$L^\# : \mathscr{H}_{1/k_0^\vee} \to \bigcup_{k \in \mathscr{K}} \mathscr{H}_k$$

is defined by

$$\begin{cases} D(L^\#) = \{v \in \mathscr{H}_{1/k_0^\vee} \mid \text{there exists } f \in \bigcup_{k\in\mathscr{K}} \mathscr{H}_k \text{ such} \\ \qquad\qquad \text{that } v(L(x,D)\phi) = f(\phi) \text{ for all } \phi \in C_0^\infty\}, \\ L^\# v = f. \end{cases} \tag{3.33}$$

PROOF. (A) Suppose that for all $\phi \in C_0^\infty$ one has

$$u_0(L(x,D)\phi) + \sum_{j=1}^{N} u_j(\gamma_0(l_j(D)\phi)) = f(\phi). \tag{3.34}$$

For all $\phi \in C_0^\infty$ we have the estimate

$$\begin{aligned} |u_0(L(x,D)\phi)| &\leqq |f(\phi)| + \sum_{j=1}^{N} |u_j(\gamma_0(l_j(D)\phi))| \\ &\leqq \|f\|_{1/k^\vee} \|\phi\|_k + \sum_{j=1}^{N} \|u_j\|_{1/h_j^\vee} \|\gamma_0(l_j(D)\phi)\|_{h_j}. \end{aligned} \tag{3.35}$$

In virtue of (2.22)

$$
\begin{aligned}
&\| \gamma_0(l_j(D)\phi) \|_{h_j}^2 \\
&= (2\pi)^{-(n-1)} \int_{\mathbf{R}^{n-1}} |\mathscr{F}_{n-1}(\gamma_0(l_j(D)\phi))(\xi')h_j(\xi')|^2 d\xi' \\
&\leqq (2\pi)^{-n} \int_{\mathbf{R}^{n-1}} \left(\int_{\mathbf{R}} |l_j(\xi', t)(\mathscr{F}\phi)(\xi', t)| \, dt \right)^2 |h_j(\xi')|^2 d\xi' \qquad (3.36) \\
&\leqq C \| \phi \|_{k_m}^2,
\end{aligned}
$$

where m is chosen so large that

$$|l_j(\xi)h_j(\xi')| \, k_{((1/2)+\varepsilon)}(\xi_n) \leqq C' k_m(\xi) \qquad \text{for all } \xi \in \mathbf{R}^n.$$

Due to (3.35) and (3.36) one sees that the linear functional $\phi \to u_0(L(x, D)\phi)$ is bounded in some space $\mathscr{H}_{k'}$. Thus one can find an element $g \in \mathscr{H}_{1/k'^{\vee}}$ such that

$$u_0(L(x, D)\phi) = g(\phi) \qquad \text{for all } \phi \in C_0^\infty,$$

that is $L^{\#} u_0 = g$.

By the relation (3.34) we obtain

$$(L^{\#} u_0 - f)(\phi) + \sum_{j=1}^{N} u_j(\gamma_0(l_j(D)\phi)) = 0 \qquad \text{for all } \phi \in C_0^\infty, \tag{3.37}$$

and then by (2.22)

$$
\begin{aligned}
&(L^{\#} u_0 - f)(\phi) + (2\pi)^{-(n-1)} \int \sum_{j=1}^{N} (\mathscr{F}_{n-1} u_j)(\xi') \overline{\mathscr{F}_{n-1}(\overline{\gamma_0(l_j(D)\phi)})}(\xi') d\xi' \\
&= (L^{\#} u_0 - f)(\phi) + (2\pi)^{-n} \int \sum_{j=1}^{N} (\mathscr{F}_{n-1} u_j)(\xi') \\
&\quad \cdot \left(\int_{\mathbf{R}} l_j(-\xi', t)(\mathscr{F}\phi)(-\xi', t) dt \right) d\xi' \qquad (3.38) \\
&= (2\pi)^{-n} \int_{\mathbf{R}^n} \mathscr{F}(L^{\#} u_0 - f)(\eta)(\mathscr{F}\phi)(-\eta) d\eta \\
&\quad + (2\pi)^{-n} \int_{\mathbf{R}^n} \sum_{j=1}^{N} (\mathscr{F}_{n-1} u_j)(\xi') l_j(-\xi', -t)(\mathscr{F}\phi)(-\xi', -t) \, dt \, d\xi' \\
&= 0.
\end{aligned}
$$

Thus we get the validity of the relation (3.32).

(B) Conversely, suppose that the relation (3.32) holds. Multiplying (3.32) by

$(2\pi)^{-n}(\mathscr{F}\phi)(-\eta)$ and integrating over $\mathbf{R}^n$ one sees by (3.38) that u lies in $D(\mathscr{L})$ and that $\mathscr{L}u = f$. Hence the proof is complete. □

3.4. Recall that the boundary operators $l_j(D)$ satisfy the condition (2.14). Then (cf. (2.24)) one sees that

$$\begin{cases} l_j(D)\phi \in C_0^\infty(\mathbf{R}^n_-) & \text{for } \phi \in C_0^\infty(\mathbf{R}^n_-) \quad \text{and} \\ l_j(D)\phi \in C_0^\infty(\mathbf{R}^n_+) & \text{for } \phi \in C_0^\infty(\mathbf{R}^n_+). \end{cases} \tag{3.39}$$

Let u be in $D(\mathscr{L})$ and let $\mathscr{L}u = f$. Then u_0 lies in $D(L^\#)$ and

$$(L^\# u_0 - f)(\phi) + \sum_{j=1}^{N} u_j(\gamma_0(l_j(D)\phi)) = 0 \qquad \text{for all } \phi \in C_0^\infty. \tag{3.40}$$

Hence one sees by (3.39)–(3.40) that

$$\operatorname{supp}(L^\# u_0 - f) \subset \mathbf{R}^n_0 := \{x \in \mathbf{R}^n \mid x_n = 0\}. \tag{3.41}$$

Lemma 3.9. *Suppose that Θ lies in $C_0^\infty(\mathbf{R})$ and define $\hat{\Theta}: \mathbf{R}^n \to \mathbf{C}$ with $\hat{\Theta}(x_1, \dots, x_n) = \Theta(x_n)$. Then for each $k \in \mathscr{K}$*

$$\| \hat{\Theta}\phi \|_k \leqq \| \Theta \|_{1,M_k^n} \| \phi \|_k \qquad \textit{for all } \phi \in C_0^\infty, \tag{3.42}$$

where the function $M_k^n: \mathbf{R} \to \mathbf{R}$ is defined by

$$M_k^n(t) = M_k(0, 0, \dots, 0, t)$$

($M_k \in \mathscr{K}$ *is defined as in* [7], p. 34).

Proof. For every $\xi = (\xi', \xi_n) \in \mathbf{R}^n$ one has

$$\mathscr{F}(\hat{\Theta}\phi)(\xi) = \int_{\mathbf{R}} \Theta(t)(\mathscr{F}'\phi)(\xi', t) e^{-i(\xi_n, t)} dt, \tag{3.43}$$

where $\mathscr{F}'$ is the partial Fourier transform defined by

$$(\mathscr{F}'\phi)(\xi', t) = \int_{\mathbf{R}^{n-1}} \phi(x', t) e^{-i(x', \xi')} dx'. \tag{3.44}$$

Hence we get

$$\mathscr{F}(\hat{\Theta}\phi)(\xi) = \int_{\mathbf{R}} \left((2\pi)^{-1} \int_{\mathbf{R}} (\mathscr{F}_1\Theta)(s) e^{i(s,t)} ds \right) (\mathscr{F}'\phi)(\xi', t) e^{-i(\xi_n,t)} dt$$

$$(3.45) \qquad = (2\pi)^{-1} \int_{\mathbf{R}} (\mathscr{F}_1\Theta)(s) \mathscr{F}_1((\mathscr{F}'\phi)(\xi', \cdot))(\xi_n - s) ds$$

$$= (2\pi)^{-1} \int_{\mathbf{R}} (\mathscr{F}_1\Theta)(s)(\mathscr{F}\phi)(\xi', \xi_n - s) ds.$$

For all $\xi = (\xi', \xi_n) \in \mathbf{R}^n$ and $s \in \mathbf{R}$ we have

$$(3.46) \qquad k(\xi) \leqq M_k(0, \ldots, 0, s) k(\xi', \xi_n - s) = M_k^n(s) k(\xi', \xi_n - s).$$

In view of (3.45)–(3.46) one obtains

$$\int_{\mathbf{R}} |\mathscr{F}(\hat{\Theta}\phi)(\xi) k(\xi)|^2 d\xi_n$$

$$(3.47) \qquad \leqq (2\pi)^{-2} \left(\int_{\mathbf{R}} |(\mathscr{F}_1\Theta)(s) M_k^n(s)| ds \right)^2 \int_{\mathbf{R}} |(\mathscr{F}\phi)(\xi', t) k(\xi', t)|^2 dt$$

$$= \| \Theta \|_{1,M_k^n}^2 \int_{\mathbf{R}} |(\mathscr{F}\phi)(\xi', t) k(\xi', t)|^2 dt.$$

and then we can conclude the validity of (3.42) by integrating both sides of (3.47) with respect to ξ' (after multiplying with $(2\pi)^{-(n-1)}$). □

Let g be in $\mathscr{H}_{k'}$ with supp $g \subset \mathbf{R}_0^n$. Furthermore, let $\theta \in C_0^\infty(\mathbf{R})$ with $\theta(x) = 1$ in the interval $[-1, 1]$. Then one sees that $\hat{\theta} g = g$. Since θt^m (with $m \in \mathbf{N}$) lies in $C_0^\infty(\mathbf{R})$ we obtain by (3.42) that

$$(3.48) \qquad \| x_n^m g \|_{k'} \leqq \| \theta t^m \|_{1,M_{k'}^n} \| g \|_{k'}.$$

Furthermore we have

Corollary 3.10. *Let $u = (u_0, u_1, \ldots, u_N)$ be in $\mathscr{H}_{1/k_0^\vee}(\mathbf{R}_+^n) \times \mathscr{B}_{1/h_1^\vee} \times \cdots \times \mathscr{B}_{1/h_N^\vee}$ and let f be in $\mathscr{H}_{1/k^\vee}(\mathbf{R}_+^n)$. Then u lies in $D(\mathscr{L})$ and $\mathscr{L}u = f$ if and only if $u_0 \in D(L^\#)$ and with each $m \in \mathbf{N}_0$ one has*

$$(3.49) \qquad \mathscr{F}(x_n^m(L^\# u_0 - f))(\xi) + (2\pi)^{-1} \sum_{j=1}^{N} l_j^{[m]}(\xi)(\mathscr{F}_{n-1}u_j)(\xi') = 0$$

a.e. $\xi = (\xi', \xi_n) \in \mathbf{R}^n$, where

$$l_j^{[m]}(\xi) := (D_n^m l_j)(-\xi).$$

Proof. (A) Due to Theorem 3.8 the validity of (3.49) (with $m = 0$) implies the validity of $\mathscr{L}u = f$.

(B) Conversely, suppose that $\mathscr{L}u = f$. Then by (3.37) we have for each $m \in \mathbf{N}_0$

$$(3.50)\quad (L^{\#}u_0 - f)(x_n^m\phi) + \sum_{j=1}^{N} u_j(\gamma_0(l_j(D)(x_n^m\phi))) = 0 \qquad \text{for all } \phi \in C_0^\infty.$$

Hence for all $\phi \in C_0^\infty$ (cf. (3.38))

$$(2\pi)^{-n}\int_{\mathbf{R}^n} \mathscr{F}(x_n^m(Lu_0 - f))(\eta)(\mathscr{F}\phi)(-\eta)$$

$$+ (2\pi)^{-n}\int_{\mathbf{R}^{n-1}}\int_{\mathbf{R}} (\mathscr{F}_{n-1}u_j)(\xi')l_j^{[m]}(\xi', t)(\mathscr{F}\phi)(-\xi', -t)\,dt\,d\xi'$$

$$(3.51)\quad = (L^{\#}u_0 - f)(x_n^m\phi)$$

$$+ (2\pi)^{-n}\sum_{j=1}^{N}\int_{\mathbf{R}^{n-1}}\int_{\mathbf{R}} (\mathscr{F}_{n-1}u_j)(\xi')l_j(-\xi', t)\mathscr{F}(x_n^m\phi)(-\xi', -t)\,dt\,d\xi'$$

$$= (L^{\#}u_0 - f)(x_n^m\phi) + \sum_{j=1}^{N} u_j(\gamma_0(l_j(D)(x_n^m\phi))) = 0,$$

since by partial integration

$$\int_{\mathbf{R}} l_j^{[m]}(\xi', t)(\mathscr{F}\phi)(-\xi', -t)dt = \int_{\mathbf{R}} l_j(-\xi', -t)D_n^m(\mathscr{F}\phi)(-\xi', -t)dt$$

$$= \int_{\mathbf{R}} l_j(-\xi', -t)\mathscr{F}(x_n^m\phi)(-\xi', -t)dt.$$

Thus the assertion follows from (3.51). □

3.5. Suppose that $u = (u_0, u_1, \ldots, u_N) \in D(\mathscr{L})$ and that $\mathscr{L}u = f$. Let $k' \in \mathscr{K}$ such that $L^{\#}u_0 - f \in \mathscr{H}_{k'}$ and choose $t \in \mathbf{N}_0$ with

$$(3.52)\qquad (1/k'^{\vee}) \leqq Ck_t.$$

Lemma 3.11. *Suppose that g lies in $\mathscr{H}_{k'}$ such that* supp $g \subset \mathbf{R}_0^n$. *Then*

$$(3.53)\qquad \mathscr{F}(x_n^t g)(\xi) = 0 \qquad a.e.\ \xi \in \mathbf{R}^n,$$

where $t \in \mathbf{N}_0$ obeys (3.52).

Proof. We obtain for all ϕ and $\psi \in C_0^\infty$ and for all $\Theta \in C_0^\infty(\mathbf{R}_-^n)$

$$|(x_n^t g)(\phi) - g(\psi)| = |g(x_n^t \phi) - g(\psi)|$$

$$\text{(3.54)} \qquad \leqq \| g \|_{k'} \| x_n^t \phi - \psi - \Theta \|_{1/k'^\vee}$$

$$\leqq C \| g \|_{k'} \| x_n^t \phi - \psi - \Theta \|_{k_t}$$

and then

$$\text{(3.55)} \qquad |(x_n^t g)(\phi) - g(\psi)| \leqq C \| g \|_{k'} \| (x_n^t \phi - \psi)\big|_{\mathbf{R}^n_+} \|_t$$

where $\| \cdot \|_t$ denotes the norm in the Sobolev space $H^t(\mathbf{R}^n_+)$ (cf. [5], p. 39). Since $x_n^t \phi\big|_{\mathbf{R}^n_+}$ lies in $H_0^t(\mathbf{R}^n_+)$ there exists a sequence $\{\psi_j\} \subset C_0^\infty(\mathbf{R}^n_+)$ such that $\| (x_n^t \phi - \psi_j)\big|_{\mathbf{R}^n_+} \|_t \to 0$ with $j \to \infty$. Because $g(\psi_j) = 0$ for each $j \in \mathbf{N}$ we obtain by (3.55) that $(x_n^t g)(\phi) = 0$ for each $\phi \in C_0^\infty$, that is, $x_n^t g = 0$. Thus the proof is complete. □

The next Theorem follows immediately from Corollary 3.10 and Lemma 3.11.

THEOREM 3.12. *Suppose that* $u = (u_0, u_1, \ldots, u_N)$ *lies in* $\mathscr{H}_{1/k_0^\vee}(\mathbf{R}^n_+) \times \mathscr{B}_{1/h_1^\vee} \times \cdots \times \mathscr{B}_{1/h_N^\vee}$ *and that* f *lies in* $\mathscr{H}_{1/k^\vee}(\mathbf{R}^n_+)$. *Then* u *lies in* $D(\mathscr{L})$ *and* $\mathscr{L}u = f$ *if and only if* $u_0 \in D(L^\#)$ *and*

$$\text{(3.56)} \qquad \operatorname{supp}(L^\# u_0 - f) \subset \mathbf{R}^n_0,$$

$$\text{(3.57)} \qquad \mathscr{F}(x_n^m (L^\# u_0 - f))(\xi) + (2\pi)^{-1} \sum_{j=1}^{N} l_j^{[m]}(\xi)(\mathscr{F}_{n-1} u_j)(\xi') = 0$$

a.e. $\xi = (\xi', \xi_n) \in \mathbf{R}^n$, *when* $m \in \{0, \ldots, t-1\}$ *and*

$$\text{(3.58)} \qquad \sum_{j=1}^{N} l_j^{[m]}(\xi)(\mathscr{F}_{n-1} u_j)(\xi') = 0 \quad a.e.\ \xi = (\xi', \xi_n) \in \mathbf{R}^n, \quad when\ m \geqq t.$$

Here $t \in \mathbf{N}_0$ *is chosen so that* (3.52) *holds*.

The condition (3.58) is interesting because it restricts the partial derivatives $l_j^{[m]}(\xi)$ of the symbols of the boundary operators $l_j(D)$.

4. On the correct solvability of the equation $\mathbf{L}^\sim U = F$

4.1. In this section we establish a criterion for the validity of the relation $R(\mathscr{L}) = \mathscr{H}_{1/k^\vee}(\mathbf{R}^n_+)$. This is utilized to obtain the coercivity of L. For the first instance we consider the existence of distributional solutions for the equation

$$\text{(4.1)} \qquad L^\# v = f \qquad \text{in } \mathbf{R}^n_+$$

with $v \in \mathscr{H}_{1/k_0^\vee}(\mathbf{R}^n_+)$ and $f \in \mathscr{H}_{1/k^\vee}(\mathbf{R}^n_+)$. More precisely, we give a criterion under which, for each $f \in \mathscr{H}_{1/k^\vee}(\mathbf{R}^n_+)$, one can find $v \in \mathscr{H}_{1/k_0^\vee}(\mathbf{R}^n_+)$ such that

$$v(L(x, D)\phi) = f(\phi) \qquad \text{for all } \phi \in C_0^\infty(\mathbf{R}^n_+). \tag{4.2}$$

We have to set many kinds of further assumptions besides the general assumptions 1°–4°. We denote $L_+^\# v = f$ when (4.2) holds.

The following theorem is an immediate consequence of the Lax–Milgram Theorem (cf. [5], p. 41).

THEOREM 4.1. *Suppose that there exist constants $c > 0$, $C > 0$, and $C' > 0$ such that for all $\psi, \phi \in C_0^\infty(\mathbf{R}^n_+)$*

$$|(\psi, L(x, D)\phi)| \leqq C \,\|\psi\|_{1/k_0} \|\phi\|_{1/k_0}, \tag{4.3}$$

$$|(\phi, L(x, D)\phi)| \geqq c \,\|\phi\|^2_{1/k_0} \qquad \textit{for all } \phi \in C_0^\infty(\mathbf{R}^n_+) \tag{4.4}$$

and

$$k \leqq C'(1/k_0). \tag{4.5}$$

Then for each $f \in \mathscr{H}_{1/k^\vee}(\mathbf{R}^n_+)$ there exists a unique $v \in \mathscr{H}_{1/k_0^\vee}(\mathbf{R}^n_+)$ such that

$$L_+^\# v = f. \tag{4.6}$$

Suppose that $L(x, D)$ satisfies the property

$$L(x, D)\phi \in C_0^\infty(\mathbf{R}^n_-) \qquad \text{for } \phi \in C_0^\infty(\mathbf{R}^n_-). \tag{4.7}$$

Let $v \in \mathscr{H}_{1/k_0^\vee}(\mathbf{R}^n_+)$ and let $f \in \mathscr{H}_{1/k^\vee}(\mathbf{R}^n_+)$ be elements such that (4.6) holds and $v \in D(L^\#)$. For all $\phi \in C_0^\infty(\mathbf{R}^n_-)$ one has

$$(L^\# v - f)(\phi) = v(L(x, D)\phi) - f(\phi) = 0, \tag{4.8}$$

and then

$$\operatorname{supp}(L^\# v - f) \subset \mathbf{R}^n_0. \tag{4.9}$$

4.2. Define a determinant $D(\xi)$ by

$$\mathrm{D}(\xi) = \begin{vmatrix} l_1(-\xi) & \cdots & l_N(-\xi) \\ \vdots & & \\ l_1^{[N-1]}(\xi) & & l_N^{[N-1]}(\xi) \end{vmatrix}. \tag{4.10}$$

Let $F_{mj}(\xi)$ be the algebraic component of $l_j^{[m]}(\xi)$ in the determinant $D(\xi)$. Assume that $D(\xi) \neq 0$ for all $\xi \in \mathbf{R}^n$. We show

Theorem 4.2. *Suppose that one can find weight functions k' and $\bar{k}_{mj} \in \mathscr{K}$ (for $(m,j) \in \{0,\ldots,N-1\} \times \{1,\ldots,N\}$) such that with K and $C > 0$*

$$(4.11) \qquad |F_{mj}(\xi)/D(\xi)| \leqq K\bar{k}_{mj}(\xi) \qquad \textit{for all } \xi \in \mathbf{R}^n,$$

$$(4.12) \qquad (1/k'^{\vee}) \leqq Ck_N,$$

for each element $f \in \mathscr{H}_{1/k^{\vee}}(\mathbf{R}^n_+)$ there exists an element $v \in \mathscr{H}_{1/k_0^{\vee}}(\mathbf{R}^n_+) \cap D(L^{\#})$ with

$$(4.13) \qquad L^{\#}v - f \in \mathscr{H}_{k'} \quad \textit{and} \quad \operatorname{supp}(L^{\#}v - f) \subset \mathbf{R}^n_0,$$

(with $\gamma > 0$)

$$(4.14) \qquad \gamma(1/h_j^{\vee}(\xi')) \leqq \left(\int_{\mathbf{R}} (k'(\xi',t)/\bar{k}_{mj}(\xi',t))^2 dt \right)^{1/2} \qquad a.e.\ \xi' \in \mathbf{R}^{n-1}$$

and

$$(4.15) \qquad l_j^{[N]}(\xi) = 0 \qquad \textit{for all } \xi \in \mathbf{R}^n, \quad j \in \{1,\ldots,N\}.$$

Then the relation

$$(4.16) \qquad R(\mathscr{L}) = \mathscr{H}_{1/k^{\vee}}(\mathbf{R}^n_+)$$

holds.

Proof. (A) Let θ be in C_0^∞ such that $\theta(0) = 1$. Define functions $\theta_l \in C_0^\infty$ by $\theta_l(x) = \theta(x/l)$. Suppose that f is in $\mathscr{H}_{1/k^{\vee}}(\mathbf{R}^n_+)$. Choose $v \in \mathscr{H}_{1/k_0^{\vee}}(\mathbf{R}^n_+)$ such that (4.13) holds. Let g be a distribution in $\mathscr{H}_{k'}$ such that $g = L^{\#}v - f$. Then one has $\operatorname{supp} g \subset \mathbf{R}^n_0$.

With each $l \in \mathbf{N}$ and $m \in \mathbf{N}_0$ the distribution $\theta_l x_n^m g$ has a compact support in $\mathbf{R}^n$. Hence its Fourier transform $\mathscr{F}(\theta_l x_n^m g)$ lies in $C^\infty(\mathbf{R}^n)$. Define for $j \in \{1,\ldots,N\}$ functions $w_j^l \in C^\infty(\mathbf{R}^n)$ by

$$(4.17) \qquad \tilde{w}_j^l(\xi) = -\sum_{m=0}^{N-1} (2\pi)(-1)^{m+j} F_{mj}(\xi)\mathscr{F}(\theta_l x_n^m g)(\xi)/D(\xi).$$

Then by the Cramer rule one has

$$(4.18) \qquad \mathscr{F}(\theta_l x_n^m g)(\xi) + \sum_{j=1}^{N} (2\pi)^{-1} l_j^{[m]}(\xi)\tilde{w}_j^l(\xi) = 0$$

for all $\xi \in \mathbf{R}^n$ and $m \in \{0,\ldots,N-1\}$.

(B) In view of (4.18) one gets

$$\sum_{j=1}^{N} l_j(-\xi)(D_n\tilde{w}_j^l)(\xi) = D_n\left(\sum_{j=1}^{N} l_j(-\xi)\tilde{w}_j^l(\xi)\right) + \sum_{j=1}^{N} l_j^{[1]}(\xi)\tilde{w}_j^l(\xi)$$

$$(4.19) \qquad = (2\pi)\mathscr{F}(\theta_l x_n g)(\xi) + \sum_{j=1}^{N} l_j^{[1]}(\xi)\tilde{w}_j^l(\xi)$$

$$= 0$$

for all $\xi \in \mathbf{R}^n$. Similarly we see by (4.18) that with each $0 \leqq m \leqq N-2$ the relation

$$(4.20) \qquad \sum_{j=1}^{N} l_j^{[m]}(\xi)(D_n\tilde{w}_j^l)(\xi) = 0 \qquad \text{for all } \xi \in \mathbf{R}^n$$

holds.

In virtue of Lemma 3.11 (with each $l \in \mathbf{N}$) we get

$$(4.21) \qquad \mathscr{F}(\theta_l x_n^N g)(\xi) = 0 \qquad \text{for all } \xi \in \mathbf{R}^n.$$

Since $l_j^{[N]}(\xi) = 0$ for all $\xi \in \mathbf{R}^n$ we obtain by (4.18)

$$\sum_{j=1}^{N} l_j^{[N-1]}(\xi)(D_n\tilde{w}_j^l)(\xi) = D_n\left(\sum_{j=1}^{N} l_j^{[N-1]}(\xi)\tilde{w}_j^l(\xi)\right) + \sum_{j=1}^{N} l_j^{[N]}(\xi)\tilde{w}_j^l(\xi)$$

$$(4.22) \qquad = (2\pi)\mathscr{F}(\theta_l x_n^N g)(\xi) + \sum_{j=1}^{N} l_j^{[N]}(\xi)\tilde{w}_j^l(\xi)$$

$$= 0$$

for all $\xi \in \mathbf{R}^n$.

Let ξ' be in $\mathbf{R}^{n-1}$. Since $D(\xi', t) \neq 0$ for all $t \in \mathbf{R}$, one sees by (4.20) and (4.22) that

$$(D_n\tilde{w}_j^l)(\xi', t) = 0 \qquad \text{for all } t \in \mathbf{R}$$

and then

$$(4.23) \qquad \tilde{w}_j^l(\xi', t) = \tilde{w}_j^l(\xi', 0) \qquad \text{for all } t \in \mathbf{R}.$$

(C) Let Θ be in C_0^∞ such that $\Theta(x) = 1$ for all $x \in [-1, 1]$. Define functions $\tilde{\tilde{w}}_j^l : \mathbf{R}^{n-1} \to \mathbf{C}$ by

$$(4.24) \qquad \tilde{\tilde{w}}_j^l(\xi') = \tilde{w}_j^l(\xi', 0).$$

Then we get by (4.14), (4.23), (4.11), (4.17) and (3.36)

$$
\begin{aligned}
&\gamma^2 |\tilde{\tilde{w}}_j^l(\xi') - \tilde{\tilde{w}}_j^i(\xi')|^2 (1/h_j^\vee(\xi'))^2 \\
&\leqq \int_{\mathbf{R}} |(\tilde{w}_j^l(\xi', 0) - \tilde{w}_j^i(\xi', 0))(k'(\xi', t)/\bar{k}_{mj}(\xi', t))|^2 dt \\
(4.25)\quad &= \int_{\mathbf{R}} |(\tilde{w}_j^l(\xi', t) - \tilde{w}_j^i(\xi', t))(k'(\xi', t)/\bar{k}_{mj}(\xi', t))|^2 dt \\
&\leqq K^2 \sum_{m=0}^{N-1} \int_{\mathbf{R}} |(\mathscr{F}(\theta_l x_n^m g)(\xi', t) - \mathscr{F}(\theta_i x_n^m g)(\xi', t))k'(\xi', t)|^2 dt \\
&= K^2 \sum_{m=0}^{N-1} \int_{\mathbf{R}} |(\mathscr{F}(\hat{\Theta} x_n^m(\theta_l g - \theta_i g))(\xi', t)k'(\xi', t)|^2 dt,
\end{aligned}
$$

where $\hat{\Theta} : \mathbf{R}^n \to \mathbf{R}$ is defined by $\hat{\Theta}(x_1, \ldots, x_n) = \Theta(x_n)$. Hence by integrating over $\mathbf{R}^{n-1}$ one sees by (3.42) that

$$
\begin{aligned}
&\gamma^2 \int_{\mathbf{R}^{n-1}} |\tilde{\tilde{w}}_j^l(\xi') - \tilde{\tilde{w}}_j^i(\xi')|^2 (1/h_j^\vee(\xi'))^2 d\xi' \\
(4.26)\quad &\leqq K^2 (2\pi)^n \sum_{m=0}^{N-1} \| \Theta t^m \|_{1, M_{k'}^n}^2 \| \theta_l g - \theta_i g \|_{k'}^2 .
\end{aligned}
$$

Since $\| \theta_l g - g \|_{k'}$ is tending to zero with $l \to \infty$, we obtain by (4.26) that there exists an element $\tilde{\tilde{w}}_j \in L_2(\mathbf{R}^{n-1})$ such that

$$
(4.27)\qquad \int_{\mathbf{R}^{n-1}} |\tilde{\tilde{w}}_j^l(\xi')/h_j^\vee(\xi')) - \tilde{\tilde{w}}_j(\xi')|^2 d\xi' \to 0 \qquad \text{with } l \to \infty.
$$

Hence one can find a subsequence $\{\tilde{\tilde{w}}_j^{l_\nu}/h_j^\vee\}$ of $\{\tilde{\tilde{w}}_j^l/h_j^\vee\}$ such that

$$
(4.28)\qquad \tilde{\tilde{w}}_j^{l_\nu}(\xi') \to h_j^\vee(\xi')\tilde{\tilde{w}}_j(\xi') \qquad \text{a.e. in } \mathbf{R}^{n-1}
$$

and

$$
(4.29)\qquad \mathscr{F}(\theta_{l_\nu} g)(\xi) \to (\mathscr{F} g)(\xi) \qquad \text{a.e. in } \mathbf{R}^n
$$

(since $\| \theta_l g - g \|_{k'} \to 0$ with $l \to \infty$).

Thus by the relation (4.18)

$$
(4.30)\qquad (\mathscr{F} g)(\xi) + \sum_{j=1}^{N} l_j^{[m]}(\xi)\tilde{\tilde{w}}_j(\xi')h_j^\vee(\xi') = 0 \qquad \text{a.e. in } \mathbf{R}^n.
$$

Since $\tilde{\tilde{w}}_j$ belongs to $L_2(\mathbf{R}^{n-1})$, the distribution generated by the function $\tilde{\tilde{w}}_j h_j^\vee$ lies in $\mathscr{S}'(\mathbf{R}^{n-1})$. Let w_j be in $\mathscr{S}'(\mathbf{R}^{n-1})$ such that $\mathscr{F}_{n-1} w_j = (2\pi)\tilde{\tilde{w}}_j h_j^\vee$. In virtue of (4.30) and Theorem 3.8 we obtain that $(v, w_1, \ldots, w_N)$ lie in $D(\mathscr{L})$ and that $\mathscr{L} u = f$. Hence $R(\mathscr{L}) = \mathscr{H}_{1/k^\vee}(\mathbf{R}_+^n)$. □

REMARK 4.3. For all $l \in \mathbf{N}$ one has (cf. the proof of the inequality (4.25))

$$\begin{aligned} &\gamma^2 |\tilde{w}_j^l(\xi')|^2 (1/h_j^\vee(\xi'))^2 \\ &\leqq K^2 \sum_{m=0}^{N-1} \int_{\mathbf{R}} |\mathscr{F}(\hat{\Theta} x_n^m \theta_l g)(\xi', t) k'(\xi', t)|^2 dt. \end{aligned} \tag{4.31}$$

Hence the distributions w_j obey

$$\begin{aligned} \| w_j \|_{1/h_j^\vee} &\leqq (K/\gamma) \left(\sum_{m=0}^{N-1} \| \Theta t^m \|_{1,M_{k'}^n}^2 \| g \|_{k'}^2 \right)^{1/2} \\ &= (K/\gamma) \left(\sum_{m=0}^{N-1} \| \Theta t^m \|_{1,M_{k'}^n}^2 \| L^\# v - f \|_{k'}^2 \right)^{1/2}. \end{aligned} \tag{4.32}$$

COROLLARY 4.4. *Suppose that*

$$\| L'(x, D)\theta \|_{1/k^\vee} \leqq C \| \theta \|_{1/k_0^\vee} \quad \textit{for all } \theta \in C_0^\infty(\mathbf{R}_+^n), \tag{4.33}$$

$$|(\psi, L(x, D)\phi)| \leqq C \| \psi \|_{1/k_0} \| \phi \|_{1/k_0} \quad \textit{for all } \psi, \phi \in C_0^\infty(\mathbf{R}_+^n), \tag{4.3}$$

$$|(\phi, L(x, D)\phi)| \geqq c \| \phi \|_{1/k_0}^2 \quad \textit{for all } \phi \in C_0^\infty(\mathbf{R}_+^n), \tag{4.4}$$

$$k \leqq C(1/k_0), \tag{4.5}$$

$$k \leqq C k_N, \tag{4.34}$$

$$D(\xi) \neq 0 \quad \textit{and} \quad |F_{mj}(\xi)/D(\xi)| \leqq K \bar{k}_{mj}(\xi) \quad \textit{for all } \xi \in \mathbf{R}^n, \tag{4.11}$$

$$\gamma(1/h_j(\xi')) \leqq \left(\int_{\mathbf{R}} (1/\bar{k}_{mj} k^\vee)(\xi', t))^2 dt \right)^2 \quad a.e. \textit{ in } \mathbf{R}^{n-1} \tag{4.14}$$

and

$$l_j^{[N]}(\xi) = 0 \quad \textit{for all } \xi \in \mathbf{R}^n, \quad j \in \{1, \ldots, N\}. \tag{4.15}$$

Then

$$R(\mathscr{L}) = \mathscr{H}_{1/k^\vee}(\mathbf{R}_+^n). \tag{4.16}$$

PROOF. In virtue of (4.3), (4.4), (4.5) and Theorem 4.1 one obtains that for each $f \in \mathscr{H}_{1/k^\vee}(\mathbf{R}_+^n)$ there exists $v \in \mathscr{H}_{1/k_0^\vee}(\mathbf{R}_+^n)$ such that $L_+^\# v = f$.

Let $\{\phi_n\}$ be a sequence in $C_0^\infty(\mathbf{R}_+^n)$ such that $\| \phi_n - v \|_{1/k_0^\vee} \to 0$ with $n \to \infty$. Then by (2.14) and (4.33) the sequence $\{L'(x, D)\phi_n\}$ lies in $C_0^\infty(\mathbf{R}_+^n)$ and it is a Cauchy sequence in the space $\mathscr{H}_{1/k^\vee}(\mathbf{R}_+^n)$. Let $h \in \mathscr{H}_{1/k^\vee}(\mathbf{R}_+^n)$ such that $\| L'(x, D)\phi_n - h \|_{1/k^\vee} \to 0$. It is easy to see that $h = L^\# v$ and then the distribution $g := L^\# v - f$ lies in $\mathscr{H}_{1/k^\vee}$ and $\operatorname{supp} g \subset \mathbf{R}_0^n$. Hence the assumptions (4.34), (4.11), (4.14) and (4.15) give in virtue of Theorem 4.2 the validity of (4.16). □

COROLLARY 4.5. *Suppose that the assumptions of Corollary 4.4 hold. Then there exists a $C>0$ such that*

$$\| \phi \|_k^+ \leqq C\left(\| L(x,D)\phi \|_{k_0}^+ + \sum_{j=1}^{N} \| \gamma_0(l_j(D) f_\phi) \|_{h_j}\right) \tag{4.35}$$

for all $\phi \in C_{(0)}^\infty(\mathbf{R}_+^n)$.

PROOF. Since by Corollary 4.4, $R(\mathscr{L}) = \mathscr{H}_{1/k^\vee}(\mathbf{R}_+^n)$, the Corollaries 3.5 and 3.7 imply that $R(\mathbf{L}^\sim)$ is closed and that $N(\mathbf{L}^\sim)=0$. Hence there exists a constant $C>0$ such that

$$\| u \|_k^+ \leqq \| \mathbf{L}^\sim u \| \qquad \text{for all } u \in D(\mathbf{L}^\sim),$$

which implies the validity of (4.35). □

4.3. Let $L(\cdot)$ be a polynomial $R^n \to \mathbf{C}$. Furthermore, suppose that there exists $k \in \mathscr{K}$ and $h_j \in \mathscr{K}(\mathbf{R}^{n-1})$ such that

$$|L(\xi)| \leqq Ck^2(\xi) \quad \text{and} \quad \operatorname{Re} L(\xi) \geqq ck^2(\xi) \qquad \text{for all } \xi \in \mathbf{R}^n, \tag{4.36}$$

$$\alpha(1+\xi_n^2)^{r/2}k_{-q}(\xi') \leqq k(\xi) \leqq \beta(1+\xi_n^2)^{r/2}h_j(\xi') \quad \text{for all } \xi = (\xi', \xi_n) \in \mathbf{R}^n \tag{4.37}$$

and

$$h_j(\xi') \leqq k_r(\xi') \qquad \text{for all } \xi' \in \mathbf{R}^{n-1} \tag{4.38}$$

with some constants $\alpha>0$, $\beta>0$, $r\in\mathbf{N}$ and $q \in \mathbf{N}$. The boundary operators $l_j(D)$ we assume to be the operators defined by $l_j(D) = D_n^{j-1}$, $j=1,\dots,r$. Then it is clear that our general assumptions 1°–4° are valid (with respect to k, $L(\xi)$ and ξ_n^{j-1}, $j=1,\dots,r$).

Let k_0 and $\tilde{k}_{mj}$ be defined by $k_0 = 1/k$ and $k_{mj}(\xi) = (1+\xi_n^2)^{r(r-1)/4}$. Then one has

$$\begin{aligned} \| L'(D)\theta \|_{1/k^\vee}^2 &= (2\pi)^{-n} \int_{\mathbf{R}^n} |L(-\xi)(\mathscr{F}\theta)(\xi)(1/k(-\xi))|^2 d\xi \\ &\leqq C \| \theta \|_{k^\vee}^2 = C \| \theta \|_{1/k_0^\vee}^2 \qquad \text{for all } \theta \in C_0^\infty, \end{aligned} \tag{4.39}$$

$$|(\psi, L(D)\phi)| = (2\pi)^{-n} \int_{\mathbf{R}^n} (\mathscr{F}\psi)(\xi)\overline{L(\xi)(\mathscr{F}\phi)(\xi)}\, d\xi$$

$$\text{(4.40)} \qquad \leqq C \,\|\psi\|_k \,\|\phi\|_k$$

$$= C\,\|\psi\|_{1/k_0}\,\|\phi\|_{1/k_0} \qquad \text{for all } \psi, \phi \in C_0^\infty,$$

$$|(\phi, L(D)\phi)| \geqq \operatorname{Re}(\phi, L(D)\phi) \geqq c\,\|\phi\|_k^2 = c\,\|\phi\|_{1/k_0}^2 \quad \text{for all } \phi \in C_0^\infty.$$
(4.41)

In addition, we have

$$l_j^{[i]}(\xi) = \begin{cases} (j-1)\cdots(j-i)\xi_n^{j-i-1}, & i \leqq j-1 \\ 0, & i > j-1 \end{cases}$$

and so

$$\text{(4.42)} \qquad D(\xi) = \begin{vmatrix} 1 & -\xi_n & \cdots & \xi_n^{r-1} \\ 0 & -1 & & -(r-1)(-\xi_n)^{r-2} \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & (-1)^{r-1}(r-1)! \end{vmatrix}$$

$$= (-1)^{r(r-1)/2}(1\cdots(r-1)!) =: a \neq 0 \qquad \text{for all } \xi \in \mathbf{R}^n$$

and

$$\text{(4.43)} \qquad |F_{mj}(\xi)| \leqq C(1+\xi_n^2)^{r(r-1)/4} = C\bar{k}_{mj}(\xi) \qquad \text{for all } \xi \in \mathbf{R}^n.$$

In virtue of (4.37) we obtain

$$\int_{\mathbf{R}} (1/(\bar{k}_{mj}k^{\vee})(\xi', t))^2 dt$$

$$\text{(4.44)} \qquad \geqq (1/C) \int_{\mathbf{R}} (1/h_j(-\xi'))^2 (1/(1+t^2)^{(r(r-1)/2)+r}) dt$$

$$=: \gamma^2 (1/h_j^{\vee}(\xi'))^2 \qquad \text{for all } \xi' \in \mathbf{R}^{n-1}.$$

Hence all the assumptions of Corollary 4.4 are valid and then

COROLLARY 4.6. *Suppose that $L(\cdot) : \mathbf{R}^n \to \mathbf{C}$ is a polynomial and that k is in $\mathscr{K}$ such that there exist q and $N \in \mathbf{N}$ with*

$$\text{(4.36)} \qquad |L(\xi)| \leqq Ck^2(\xi) \quad \textit{and} \quad \operatorname{Re} L(\xi) \geqq ck^2(\xi),$$

$$\text{(4.37)} \qquad \alpha(1+\xi_n^2)^{r/2} k_{-q}(\xi') \leqq k(\xi) \leqq \beta(1+\xi_n^2)^{r/2} h_j(\xi')$$

and

$$h_j(\xi') \leqq k_r(\xi') \tag{4.38}$$

for all $\xi = (\xi', \xi_n) \in \mathbf{R}^n$. *Then there exists a constant* $C > 0$ *such that*

$$\| \phi \|_k^+ \leqq C\left(\| L(D)\phi \|_{1/k}^+ + \sum_{j=1}^{r} \| \gamma_0(D_n^{j-1} f_\phi) \|_{h_j} \right) \tag{4.45}$$

for all $\phi \in C_{(0)}^\infty(\mathbf{R}_+^n)$.

In the case of the Laplace operator $L(D) = D_1^2 + D_2^2$ the weight functions k and h_j can be chosen to be $k(\xi) = (1 + |\xi|^2)^{1/2}$ and $h_1(\xi') = (1 + |\xi'|^2)^{1/2}$.

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